SEMI-CLASSICAL ANALYSIS OF OCEANIC FLOWS

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Abstract. In these notes we present results of [5] and [8] in which the propagation of waves is studied, in a model describing the movement of oceans in large geographical zones. We consider a shallow water flow subject to strong rotation and linearized around an inhomogeneous stationary profile, and we prove that the underlying system of PDEs can be diagonalized microlocally: the three linear propagators thus constructed correspond to particular types of waves, namely two Poincaré and one Rossby wave. We show how Mourre estimates allow to obtain the dispersion of Poincaré waves; in the case when the stationary profile is zonal we prove by ODE techniques that for initial data microlocalized in some codimension one set, Rossby waves are trapped for all times.

1. Introduction

The aim of these notes is to present results of [5] and [8], in which the long time propagation of waves induced by a rotating, shallow water model is analyzed.

1.1. The model.

The ocean is considered in this (toy) model as an incompressible, inviscid fluid with free surface submitted to gravitation and wind forcing, and we further make the following classical assumptions: we assume that the density of the fluid is homogeneous (meaning that the density ρ is equal to a constant ρ_0), that the pressure law is given by the hydrostatic approximation $p = \rho_0 g z$, and that the motion is essentially horizontal and does not depend on the vertical coordinate. This leads to the so-called shallow water approximation.

For the sake of simplicity, the effects of the interaction with the boundaries are not discussed and the model is purely horizontal with the longitude x_1 and the latitude x_2 both in $\mathbf{R}$.

The evolution of the water height h and velocity v is then governed by the shallow-water equations with Coriolis force

$$\begin{aligned} \partial_t(\rho_0 h) + \nabla\cdot(\rho_0 h v) &= 0 \\ \partial_t(\rho_0 h v) + \nabla\cdot(\rho_0 h v\otimes v) + \omega(\rho_0 h v)^\perp + \rho_0 g h \nabla h &= \rho_0 h \tau \end{aligned} \tag{1.1}$$

where ω denotes the vertical component of the Earth rotation vector Ω, $v^\perp := (-v_2, v_1)$, g is the gravity and τ is the stationary forcing responsible for the macroscopic flow. The vertical component of the Earth rotation is therefore $\Omega\sin(x_2/R)$, where R is the radius of the Earth; note that it is classical in the physical literature to consider the linearization of ω (known as the betaplane approximation) $\omega(x_2) = \Omega x_2/R$. We consider general functions ω in the sequel, with some restrictions that are made precise later.

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We consider small fluctuations (η, u) around the stationary solution $(\bar h, \bar v)$ satisfying

$$\bar h = \text{constant}, \quad \nabla \cdot (\bar v \otimes \bar v) + \omega \bar v^\perp = \tau, \quad \operatorname{div} \bar v = 0\,.$$

Let us adimensionalize the previous system. We define $b := \omega/|\Omega|$, and we consider typical time, length and velocity scales $t_0 \sim 10^6\,\mathrm{s}\ (\sim 0,38\,\text{months})$, $\ell_0 \sim 10^4\,\mathrm{km}$ and $v_0 \sim 0.1\mathrm{ms}^{-1}$. We also consider typical height and velocity fluctuations $\delta h = (h - \bar h)/\eta \ \sim 1\,\mathrm{m}$ and $u = (v - \bar v)/v_0 = v/v_0 - \bar u$. Finally we define a small parameter $\varepsilon \sim 10^{-1}$ (actually of the size of Fr^2 and $\mathrm{Ro}^{\frac12}$ where Fr is the Froude number and Ro the Rossby number, measuring respectively the influence of gravity and of the Coriolis force). After some computations we come up with the following system:

$$\begin{aligned} \partial_t \eta + \frac{1}{\varepsilon}\nabla \cdot u + \bar u \cdot \nabla \eta + \varepsilon^2 \nabla \cdot (\eta u) = 0\,,\\ \partial_t u + \frac{1}{\varepsilon^2} b u^\perp + \frac{1}{\varepsilon}\nabla \eta + \bar u \cdot \nabla u + u \cdot \nabla \bar u + \varepsilon^2 u \cdot \nabla u = 0\,. \end{aligned} \tag{1.2}$$

Defining the sound speed u_0 by

$$\eta = \big[(1 + \varepsilon^3 u_0/2)^2 - 1\big]/\varepsilon^3\,,$$

we obtain that (1.2) is equivalent to

$$\varepsilon^2 \partial_t U + A(x, \varepsilon D, \varepsilon) U + \varepsilon^3 Q(U) = 0\,, \qquad U = (u_0, u_1, u_2) \tag{1.3}$$

where $A(x, \varepsilon D, \varepsilon)$ is the linear propagator

$$A(x, \varepsilon D, \varepsilon) := i \begin{pmatrix} \varepsilon \bar u \cdot \varepsilon \nabla & \varepsilon \partial_1 & \varepsilon \partial_2 \\ \varepsilon \partial_1 & \varepsilon \bar u \cdot \varepsilon \nabla + \varepsilon^2 \partial_1 \bar u_1 & -b + \varepsilon^2 \partial_2 \bar u_1 \\ \varepsilon \partial_2 & b + \varepsilon^2 \partial_1 \bar u_2 & \varepsilon \bar u \cdot \varepsilon \nabla + \varepsilon^2 \partial_2 \bar u_2 \end{pmatrix} \tag{1.4}$$

and $Q(U) := S_1(U)\varepsilon\partial_1 U + S_2(U)\varepsilon\partial_2 U$ with

$$S_1(U) := \begin{pmatrix} u_1 & \frac12 u_0 & 0 \\ \frac12 u_0 & u_1 & 0 \\ 0 & 0 & u_1 \end{pmatrix} \quad \text{and} \quad S_2(U) := \begin{pmatrix} u_2 & 0 & \frac12 u_0 \\ 0 & u_2 & 0 \\ \frac12 u_0 & 0 & u_2 \end{pmatrix}. \tag{1.5}$$

We shall assume throughout the paper that b is smooth, with a symbol-like behaviour: for all $\alpha \in \mathbf{N}$, there is a constant C_α such that for all $x_2 \in \mathbf{R}$,

$$|b^{(\alpha)}(x_2)| \le C_\alpha \big(1 + b^2(x_2)\big)^{\frac12}. \tag{1.6}$$

We shall further assume that

$$\lim_{|x_2| \to \infty} b^2(x_2) = \infty,$$

and that b^2 has only non degenerate critical points. We shall also assume that $\bar u$ is a smooth, compactly supported function.

We shall finally suppose that the initial data is microlocalized (see Appendix A for definitions) in some compact set $\mathcal{C}$ of $T^*\mathbf{R}^2$ satisfying

$$\mathcal{C} \cap \{\xi_1 = 0\} = \emptyset\,. \tag{1.7}$$

In the following, to simplify some formulations, we shall denote by $(\mu)\mathrm{Supp}_\star f$ the projection of the (micro)support of f onto the $\star = 0$ axis, where $\star$ represents an element of $\{x_1, x_2, \xi_1, \xi_2\}$.

Because of the specific form of the initial data, involving fast oscillations with respect to x, we introduce semi-classical Sobolev spaces

$$H^s_\varepsilon = \{U \in L^2 \,/\, \|U\|_{H^s_\varepsilon} < +\infty\} \quad \text{with} \quad \|U\|^2_{H^s_\varepsilon} = \sum_{|k|\leq s} \|(\varepsilon\nabla)^k U\|^2_{L^2}\,.$$

A classical result based on the Sobolev embedding

$$\|\varepsilon\nabla U\|_{L^\infty} \leq \frac{C}{\varepsilon}\|\nabla U\|_{H^s_\varepsilon} \text{ for any } s > 1\,,$$

implies that (1.2) has a unique local solution $U_\varepsilon \in L^\infty([0,T_\varepsilon), H^{s+1}_\varepsilon)$. Note that the life span of U_ε depends a priori on ε. One of the results proved in these notes is existence on an ε-independent time interval (see Theorem 6 in Section 5), assuming the initial data is bounded in a weighted semi-classical Sobolev space, adapted to the linear propagator (in the spirit of [7]):

$$W^s_\varepsilon := \Big\{ f \in L^2(\mathbf{R}^2) \,/\, (1-\varepsilon^2\partial_1^2)^{\frac{s}{2}}(1-\varepsilon^2\partial_2^2 + b^2(x_2))^{\frac{s}{2}} f \in L^2(\mathbf{R}^2) \Big\}\,. \tag{1.8}$$

But most of the analysis will actually be carried out on the linear system

$$\varepsilon^2\partial_t V + A(x,\varepsilon D,\varepsilon)V = 0\,. \tag{1.9}$$

The structure of these notes is the following.

- In Section 2 we prove an abstract diagonalization result (Theorem 2) on systems of the form (1.9), when the principal symbol matrix of $A(x,\varepsilon D,\varepsilon)$ is diagonalizable with eigenvalues which do not cross. We apply the general result to the specific case of (1.4), which allows to compute Poincaré and Rossby operators (Theorem 1).
- In Section 3 we prove that Poincaré waves exit any compact set for any positive time (Theorem 3). The proof relies on Mourre estimates. We also provide a different proof in the case when $\bar u$ is a shear flow, which relies on a semi-explicit representation of the solution. Though less general than the first one, we feel this proof can have some interest in nonlinear applications for instance, due to its more explicit form.
- Section 4 is devoted to the study of Rossby waves. It is proved that Rossby waves stay confined for all times in the latitude (x_2) direction (Theorem 4). In the longitude (x_1) direction such a result is only obtained in the special case of a shear flow (see Theorem 5). To prove the result we study the integral curves of the associated Hamiltonian.
- Finally in Section 5 we prove that the life span of the nonlinear system (1.3) is uniformly bounded from below, and check that the nonlinear solution stays close to the linear one. The method relies on the construction of almost commuting vector fields, which is possible here due to the semi-classical nature of the problem. That explains the introduction of W^s_ε in (1.8).
- We have gathered in an appendix all useful results in microlocal and semiclassical analysis. The reader is invited to consult the appendix for definitions and notations used throughout the text.

Remark 1.1. *We study here a very particular situation when the initial data is localized in a very small region of space, and has a compact frequency support – this allows to apply semi-classical methods and have an effective way of studying the propagation of waves. Note that many studies have been devoted to other situations where the initial data does not present such localization properties. We refer for instance to* [1], [2], [3], [4], [7], [9], [10], [17], [18],

[19] *and the references therein for such studies in the context of rotating fluids. We refer also to* [15], [16], [20], *and* [21] *for references in the Phyiscal litterature.*

2. Diagonalization

2.1. **Statement of the theorem.**

In this section we shall prove the following result.

Theorem 1 (Rossby and Poincaré propagators)**.** *Let $\mathbf{v}_{\varepsilon,0}$ be a family of initial data, microlocalized in a compact set $\mathcal{C}$ satisfying Assumption* (1.7)*. For any parameter $\varepsilon > 0$, denote by $\mathbf{v}_\varepsilon$ the associate solution to* (1.9)*. Then for all $t \geq 0$ one can write $\mathbf{v}_\varepsilon(t)$ as the sum of a "Rossby" vector field and two "Poincaré" vector fields: $\mathbf{v}_\varepsilon(t) = \mathbf{v}_\varepsilon^R(t) + \mathbf{v}_\varepsilon^+(t) + \mathbf{v}_\varepsilon^-(t)$, satisfying linear equations*

$$i\varepsilon\partial_t \mathbf{v}_\varepsilon^R = T_R \mathbf{v}_\varepsilon^R, \quad i\varepsilon^2\partial_t \mathbf{v}_\varepsilon^\pm = T_\pm \mathbf{v}_\varepsilon^\pm\,,$$

where the principal symbol of each operator is given by

$$\sigma_p(T_R) = \frac{\xi_1 b'}{\xi_1^2+\xi_2^2+b^2(x_2)} + \bar u\cdot\xi \quad \textit{and} \quad \sigma_p(T_\pm) = \pm\sqrt{\xi_1^2+\xi_2^2+b^2(x_2)}\,.$$

This theorem is a consequence of general diagonalization result stated and proved in the coming paragraph.

2.2. **A general diagonalization theorem.**

Theorem 2 (Diagonalization)**.** *Let $\mathcal{K}$ be a compact subset of $\mathbf{R}^{2d}$, and consider a $N\times N$ hermitian pseudodifferential matrix $A_\varepsilon = A(x,\varepsilon D,\varepsilon)$, supported in $\mathcal{K}$. Assume that*

- *the (matrix) principal symbol of $A(x,\varepsilon D,0)$, denoted by $\mathcal{A}_0$, is diagonalizable, in the sense that there are some unitary and diagonal matrices of symbols, $\mathcal{U}$ and $\mathcal{D}$, such that $\mathcal{U}^{-1}\mathcal{A}_0\mathcal{U} = \mathcal{D}$,*
- *the eigenvalues $(\delta_1(x,\xi),\ldots,\delta_N(x,\xi))$ satisfy*

$$\forall i\neq j, \quad \inf_{(x,\xi)\in\mathcal{K}} |\delta_i(x,\xi)-\delta_j(x,\xi)| \geq C > 0\,. \tag{2.1}$$

Then there exists a family of unitary and diagonal pseudodifferential operators V_ε and D_ε supported in $\mathcal{K}$, such that:

$$V_\varepsilon^* A_\varepsilon V_\varepsilon = D_\varepsilon + O(\varepsilon^\infty), \quad V_\varepsilon^* V_\varepsilon = I + O(\varepsilon^\infty)\,. \tag{2.2}$$

Moreover one has

$$D_\varepsilon = D_0 + \varepsilon D_1 + O(\varepsilon^2)\,, \tag{2.3}$$

where $D_0 = \mathrm{Op}_\varepsilon^W(\mathcal{D})$ and the principal symbol of D_1 is given by

$$\mathcal{D}_1 = \sigma_p(D_1) = \textit{ diag } \left(\sigma_p(\widetilde\Delta_1 - \frac{D_0 I_1 + I_1 D_0}{2})\right)$$

with the notations

$$\widetilde\Delta_1 = \frac{1}{\varepsilon}\left(\mathrm{Op}_\varepsilon^W(\mathcal{U}^*)A_\varepsilon \mathrm{Op}_\varepsilon^W(\mathcal{U}) - D_0\right), \quad I_1 = \frac{1}{\varepsilon}\left(\mathrm{Op}_\varepsilon^W(\mathcal{U}^*)\mathrm{Op}_\varepsilon^W(\mathcal{U}) - I\right)\,. \tag{2.4}$$

More explicitly, let us denote by $a_{ij}(x,\xi)$ the matrix elements of $\mathcal{A}_1$, subsymbol of $A(x,\varepsilon D,\varepsilon)$ defined by $\mathcal{A}_1 := \sigma_p(\partial_\varepsilon A)$ and by $u_{nj}(x,\xi)$, $j=1\dots d$, the coordinates of any unit eigenvector of $\mathcal{A}_0(x,\xi)$ of eigenvalue $\delta_n(x,\xi)$. We have

(2.5)

$$(\mathcal{D}_1)_{nn} = \sum_{j,k=1\dots d}\left(\Im\left(\overline{u_{jn}}\{a_{jk},u_{kn}\}\right) + \frac{a_{jk}\{\overline{u_{jn}},u_{kn}\}}{2i}\right) + (\mathcal{U}^*\mathcal{A}_1\mathcal{U})_{nn}) + \frac{1}{2i}\sum_{j=1}^{d}\delta_n\{u_{jn},\overline{u_{jn}}\},$$

where $\{f,g\} := \nabla_\xi f\nabla_x g - \nabla_x f\nabla_\xi g$ is the Poisson bracket on $T^*\mathbf{R}^d$.

Here and in all the sequel, we say that a pseudo-differential operator V is unitary if it satisfies

$$V^*V = I + O(\varepsilon^\infty).$$

The proof is presented below: we only give the formal construction and we leave it to the reader to check that the symbols of the various operators formally constructed are indeed symbols. Section 2.3 is devoted to the case of the matrix given by (1.4).

The proof of Theorem 2 is a combination of semiclassical and perturbation methods. Let us start by defining $U_0 := \mathrm{Op}_\varepsilon^W(\mathcal{U})$. Elementary properties of the Weyl quantization imply then that

$$U_0^* A_\varepsilon U_0 = D_0 + O(\varepsilon)\,.$$

The following lemma shows that one can construct a unitary pseudodifferential operator U_∞ such that

$$U_\infty^* A_\varepsilon U_\infty = D_0 + O(\varepsilon)\,.$$

Lemma 2.1. *Let U be a pseudodifferential matrix such that $U^*U = I + \varepsilon I_1$, where I is the identity. Then one can find $V \sim \sum_{k=0}^{\infty} \varepsilon^k V_k$ such that $(U+\varepsilon V)^*(U+\varepsilon V) = I + O(\varepsilon^\infty)$.*

Proof. Let us denote $V_0 := -\frac{1}{2}UI_1$. On easily checks that $(U+\varepsilon V_0)^*(U+\varepsilon V_0) = I + O(\varepsilon^2)$. Indeed

$$\begin{aligned}(U+\varepsilon V_0)^*(U+\varepsilon V_0) &= U^*U - \frac{\varepsilon}{2}(I_1U^*U + U^*UI_1) + O(\varepsilon^2)\\ &= I + \varepsilon I_1 - \varepsilon I_1 + O(\varepsilon^2)\,.\end{aligned}$$

Then one concludes by iteration. □

That lemma allows to define the pseudo-differential operator of (semiclassical) order 0

$$\Delta_1 = \frac{1}{\varepsilon}\left(U_\infty^* A_\varepsilon U_\infty - D_0\right)\,,$$

where U_∞ is a unitary operator. Now our aim is to find a unitary operator V_∞ (up to $O(\varepsilon^\infty)$) such that

$$(U_\infty V_\infty)^* A_\varepsilon (U_\infty V_\infty) = D_\infty + O(\varepsilon^\infty)\,,$$

where $D_\infty = D_0 + \varepsilon D_1 + \dots$ is a diagonal matrix satisfying the conclusions of the theorem. We shall write $V_\infty = e^{i\varepsilon W}$, with W selfadjoint (so V_∞ thus constructed is automatically unitary).

We look for W under the form $W \sim \sum_0^\infty \varepsilon^k W_k$, and compute the W_k recursively. Since

$$V_\infty^*(D_0+\varepsilon\Delta_1)V_\infty = (D_0+\varepsilon\Delta_1) + i\varepsilon[(D_0+\varepsilon\Delta_1),W] + \frac{(i\varepsilon)^2}{2}[[(D_0+\varepsilon\Delta_1),W],W]+\dots$$

we see that, if W_1 satisfies

$$i[D_0,W_1]+\Delta_1 = D_1+O(\varepsilon),\ D_1 \text{ diagonal}\,, \tag{2.6}$$

then we have that

$$e^{-i\varepsilon W_1}(D_0+\varepsilon\Delta_1)e^{i\varepsilon W_1} = D_0+\varepsilon D_1+\varepsilon^2\Delta_2\,, \tag{2.7}$$

where Δ_2 is a zero order pseudodifferential operator. The following lemma is a typical normal form type result, and is crucial for the following.

Lemma 2.2. *Let D_0 be a diagonal pseudodifferential matrix whose principal symbol $\mathcal{D}_0$ has a spectrum satisfying (2.1) and let Δ_1 be a pseudodifferential matrix. Then there exist two pseudodifferential matrices W and D_1, with D_1 diagonal, such that:*

$$[D_0,W]+\Delta_1 = D_1+\varepsilon\widetilde{\Delta}_2\,, \tag{2.8}$$

where $\widetilde{\Delta}_2$ is a pseudodifferential matrix of order 0. Moreover the principal symbol of D_1 is the diagonal part of the principal symbol of Δ_1: we have $\sigma_p(D_1) = diag\,\sigma_p(\Delta_1)$.

Proof. By the non degeneracy condition of the spectrum of $\mathcal{D}_0$ we find that there exists a matrix $\mathcal{W}_0$ and a diagonal one $\mathcal{D}_1$ such that $[\mathcal{D}_0,\mathcal{W}_0]+\mathcal{D}_{1,0} = \mathcal{D}_1$, where $\mathcal{D}_{1,0}$ is the principal symbol of Δ_1. Indeed it is enough to take $\mathcal{D}_1$ as the diagonal part of $\mathcal{D}_{1,0}$ and

$$(\mathcal{W}_0(x,\xi))_{i,j} = \frac{(\mathcal{D}_{1,0}(x,\xi))_{i,j}}{\delta_i(x,\xi)-\delta_j(x,\xi)} \tag{2.9}$$

and notice that the Weyl quantization of $\mathcal{W}_0$ satisfies (2.8). The lemma is proved. □

By Lemma 2.2 we know that there exists W_1 satisfying (2.6). Writing

$$e^{-i\varepsilon W_1}(D_0+\varepsilon\Delta_1)e^{i\varepsilon W_1} = D_0+\varepsilon(\Delta_1+[D_0,W_1])+\varepsilon^2(\Delta_2-\tilde{\Delta}_2),$$

we get immediately (2.7). It is easy to get convinced that all the W_k will satisfy recursively an equation of the form $[D_0,W_k]+\Delta_k = D_k+O(\varepsilon)$, which can be solved by Lemma 2.2. In order to derive (2.5) we have to compute the subprincipal symbol of the diagonal part of the right-hand side of (2.4), that is, for each $n=1\dots d$,

$$\sum_{jk}\mathrm{Op}_\varepsilon^W(\overline{\mathcal{U}_{jn}})\mathrm{Op}_\varepsilon^W((\mathcal{A}_0+\varepsilon\mathcal{A}_1)_{jk})\mathrm{Op}_\varepsilon^W(\mathcal{U}_{kn}) - \frac{1}{2i}\sum_{j=1}^d \delta_n\{\overline{u_{jn}},u_{jn}\}\,,$$

since $\mathcal{U}$ is unitary. The term $\varepsilon\mathcal{A}_1$ is obviously responsible for the second term in the right-hand side of (2.5). Using the distributivity of the Poisson bracket, we get the following expression for the first one:

$$\sum_{jk}\frac{1}{2i}\left(\{\overline{\mathcal{U}_{jn}},(\mathcal{A}_0)_{jk}\mathcal{U}_{kn}\}+\overline{\mathcal{U}_{jn}}\{(\mathcal{A}_0)_{jk},\mathcal{U}_{kn}\}\right)$$

$$=\sum_{jk}\frac{1}{2i}\left(\overline{\mathcal{U}_{jn}}\{(\mathcal{A}_0)_{jk},\mathcal{U}_{kn}\}+(\mathcal{A}_0)_{jk}\{\overline{\mathcal{U}_{jn}},\mathcal{U}_{kn}\}+\mathcal{U}_{kn}\{\overline{\mathcal{U}_{jn}},(\mathcal{A}_0)_{jk}\}\right)\,.$$

Interverting j and k in half of the terms and noticing that, since $\mathcal{A}_0$ is Hermitian, $(\mathcal{A}_0)_{jk} = \overline{(\mathcal{A}_0)_{kj}}$, we get easily (2.5).

2.3. **The Rossby-Poincaré case.**

In the case of oceanic waves we compute

$$\mathcal{A}_0(x,\xi) := \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & 0 & -ib \\ \xi_2 & ib & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_1(x,\xi) := \begin{pmatrix} \bar{u}\cdot\xi & 0 & 0 \\ 0 & \bar{u}\cdot\xi & 0 \\ 0 & 0 & \bar{u}\cdot\xi \end{pmatrix}.$$

A straightforward computation shows that the spectrum of $\mathcal{A}_0$ is

$$\left\{ 0, \sqrt{\xi_1^2+\xi_2^2+b^2(x_2)}, -\sqrt{\xi_1^2+\xi_2^2+b^2(x_2)} \right\}.$$

2.3.1. *Microlocalization.*

The three eigenvalues of $\mathcal{A}_0$ are separated if and only if

$$\xi_1^2+\xi_2^2+b^2(x_2) \neq 0.$$

Therefore, considering a compact subset $\mathcal{K}$ of $\mathbf{R}^4$ such that

$$\mathcal{K} \cap \{(x_1,x_2,\xi_1,\xi_2) \,/\, \xi_1^2+\xi_2^2+b^2(x_2) = 0\} = \emptyset$$

ensures that

- the eigenvalues do not cross, so that it is possible to get a unitary diagonalizing matrix with regular entries;
- the non degeneracy condition (2.1) is satisfied.

In other words, $A(x,\varepsilon D,\varepsilon)$ satisfies the assumptions of Theorem 2 provided that one considers only its action on vector fields which are suitably microlocalized. We assume of course that this microlocalization condition is satisfied by the initial datum, or actually the more restrictive condition (1.7). Furthermore, we shall prove in the next two sections that the propagation by the scalar operators $T_\pm$ and T_R (to be defined now) preserves this suitable microlocalization, thus justifying a posteriori the diagonalization procedure for all times.

2.3.2. *Computation of the Poincaré and Rossby Hamiltonians.*

The above computations show that one can define the two Poincaré Hamiltonians as follows:

$$\tau_\pm := \pm\sqrt{\xi_1^2+\xi_2^2+b^2(x_2)}$$

and we shall denote the associate operator constructed via Theorem 2 by $T_\pm$.

Now let us consider the Rossby Hamiltonian. In all this paragraph, for the sake of readability, we shall denote

$$\langle\xi\rangle_b := \sqrt{\xi_1^2+\xi_2^2+b^2(x_2)}\,.$$

An easy computation shows that a (normalized) eigenvector of $\mathcal{A}_0(x,\xi)$ of zero eigenvalue is

$$u_0 = \frac{1}{\langle\xi\rangle_b}\begin{pmatrix} b \\ i\xi_2 \\ -i\xi_1 \end{pmatrix}.$$

By Theorem 2, the Rossby Hamiltonian is then given by the formula

$$\tau_R = \sum_{j,k=1\dots 3} \left(\Im\left(\overline{u_{j0}}\{a_{jk}, u_{k0}\}\right) + \frac{a_{jk}\{\overline{u_{j0}}, u_{k0}\}}{2i} \right) + \sum_{j,k=1\dots 3} (\mathcal{A}_1)_{jk} \overline{u_{j0}} u_{k0} \,. \tag{2.10}$$

The contribution of the first term in the parenthesis in (2.10) is

$$\begin{aligned}
&\sum_{j,k=1\dots 3} \left(\overline{u_{j0}}\{a_{jk}, u_{k0}\}\right) \\
&\quad = \frac{b}{\langle\xi\rangle_b}\left\{\xi_2, \frac{-i\xi_1}{\langle\xi\rangle_b}\right\} + \frac{i\xi_2}{\langle\xi\rangle_b}\left\{ib, \frac{-i\xi_1}{\langle\xi\rangle_b}\right\} + \frac{i\xi_1}{\langle\xi\rangle_b}\left(\left\{\xi_2, \frac{b}{\langle\xi\rangle_b}\right\} + \left\{ib, \frac{i\xi_2}{\langle\xi\rangle_b}\right\}\right) \\
&\quad = \frac{-ib\xi_1}{\langle\xi\rangle_b}\partial_{x_2}\frac{1}{\langle\xi\rangle_b} - \frac{i\xi_2\xi_1 b'}{\langle\xi\rangle_b}\partial_{\xi_2}\frac{1}{\langle\xi\rangle_b} + \frac{i\xi_1}{\langle\xi\rangle_b}\partial_{x_2}\frac{b}{\langle\xi\rangle_b} + \frac{i\xi_1 b'}{\langle\xi\rangle_b}\partial_{\xi_2}\frac{\xi_2}{\langle\xi\rangle_b} \\
&\quad = \frac{2i\xi_1 b'}{\langle\xi\rangle_b^2} \,\cdot
\end{aligned}$$

Using the distributivity of the Poisson brackets, we get the contribution of the second term in a very similar way

$$\begin{aligned}
&\sum_{j,k=1\dots 3} \frac{a_{jk}\{\overline{u_{j0}}, u_{k0}\}}{2} \\
&\quad = \xi_1\left\{\frac{b}{\langle\xi\rangle_b}, \frac{i\xi_2}{\langle\xi\rangle_b}\right\} - \xi_2\left\{\frac{b}{\langle\xi\rangle_b}, \frac{i\xi_1}{\langle\xi\rangle_b}\right\} + ib\left\{\frac{i\xi_1}{\langle\xi\rangle_b}, \frac{i\xi_2}{\langle\xi\rangle_b}\right\} \\
&\quad = \xi_1\left(\frac{ib}{\langle\xi\rangle_b}\left\{\frac{1}{\langle\xi\rangle_b}, \xi_2\right\} + \frac{i\xi_2}{\langle\xi\rangle_b}\left\{b, \frac{1}{\xi_b}\right\} + \frac{i}{\langle\xi\rangle_b^2}\{b, \xi_2\}\right) - \frac{i\xi_2\xi_1}{\langle\xi\rangle_b}\left\{b, \frac{1}{\langle\xi\rangle_b}\right\} - \frac{ib\xi_1}{\langle\xi\rangle_b}\left\{\frac{1}{\langle\xi\rangle_b}, \xi_2\right\} \\
&\quad = \frac{-ib'\xi_1}{\langle\xi\rangle_b^2}\cdot
\end{aligned}$$

The computation of the second term of the right hand side of (2.10) is trivial since $\mathcal{A}_1$ is a multiple of the identity. Adding the two previous expressions we get finally

$$\tau_R = \frac{\xi_1 b'}{\xi_1^2 + \xi_2^2 + b(x_2)^2} + \bar{u}\cdot\xi$$

and the associate operator will be denoted by T_R.

Remark 2.3. *Since the elementary steps of the diagonalization process use only multiplications, computations of subprincipal symbols and solving normal form equations, all the subsymbols of T_R and $T_\pm$ depend on x_1 only through $\bar{u}$ and its derivatives.*

3. Dispersion of Poincaré waves

The goal of this section is to prove the following result.

Theorem 3 (Dispersion of Poincaré waves)**.** *Let $\mathbf{v}_{\varepsilon,0}$ be a family of initial data, microlocalized in a compact set $\mathcal{C}$ satisfying Assumption* (1.7)*. For any parameter $\varepsilon > 0$, denote by $\mathbf{v}_\varepsilon^P$ the Poincaré component of the solution $\mathbf{v}_\varepsilon$ to* (1.9) *constructed in Theorem* 1*. Then for any compact set Ω in $\mathbf{R}^2$, one has*

$$\forall t > 0, \quad \|\mathbf{v}_\varepsilon^P(t)\|_{L^2(\Omega)} = O(\varepsilon^\infty)\,.$$

In Theorem 1 we constructed two linear operators, called $T_\pm$, whose principal symbols are

$$\tau_\pm = \pm\sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}\,.$$

We now want to study the propagation equation associated to those operators, namely the linear equation in $\mathbf{R} \times \mathbf{R}^2$

$$i\varepsilon^2 \partial_t \varphi_\pm = T_\pm \varphi_\pm, \quad \varphi_{\pm|t=0} = \varphi_\pm^0 \tag{3.1}$$

where $\varphi_\pm^0$ are microlocalized in a compact set $\mathcal{C}$ satisfying Assumption (1.7). Before studying that equation we need to check that it makes sense, since a priori $T_\pm$ is only defined on vector fields microlocalized on such a compact set. This is achieved in the coming section, where we check that the separation of eigenvalues (2.1) required in the statement of Theorem 2 holds because $\sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$ remains bounded away from zero during the propagation.

Then we shall show that the solutions to these equations exit any compact set in finite time.

3.1. **Microlocalization.** Let us prove the following result, which allows to make sense of Equation (3.1) for all times.

Proposition 3.1. *Under the assumptions of Theorem* 3*, the operators* $T_\pm$ *are self-adjoint, and the function* $\varphi(t) = e^{i\frac{t}{\varepsilon^2}T_\pm}\varphi_\pm^0$ *are such that* $\mu Supp \varphi_\pm(t)$ *satisfies* (1.7) *for all times.*

Proof. The proof of that result relies on a spectral argument. Due to the form of the principal symbols of $T_\pm$ recalled above, the operators $T_\pm$ are self-adjoint and for each fixed ξ_1 have discrete spectrum. We can therefore define two families $(\psi_{n,\mu}^\pm)_{n\in\mathbf{N},\mu\in\mathbf{R}}$ of pseudo-eigenvectors of $T_\pm$ in $L^2(\mathbf{R}^2)$ and eigenvalues $\lambda_{n,\mu}^\pm$ such that if the initial data writes

$$\varphi_\pm^0(x) = \int \sum_n c_{n,\mu}^{\pm,0} \psi_{n,\mu}^\pm(x)\, d\mu\,,$$

then

$$\varphi_\pm(t,x) = \int \sum_n e^{i\frac{\lambda_{n,\mu}^\pm t}{\varepsilon^2}} c_{n,\mu}^{\pm,0} \psi_{n,\mu}^\pm(x)\, d\mu\,.$$

Since the eigenfunctions $\psi_n^\pm$ are microlocalized on the energy surfaces of the Poincaré Hamiltonians, the result follows. □

3.2. **Dispersion.** In this paragraph we shall prove the dispersion result. The strategy is the following. In Section 3.2.1 we prove using semi-classical analysis that for a very short time, the solutions to (3.1) remain microlocalized in a compact set satisfying assumption (1.7), and such that $\mu\mathrm{Supp}_{x_1}\varphi_\pm$ become disjoint from $\mathrm{Supp}_{x_1}\bar{u}$. Section 3.2.2 is then devoted to the long-time behaviour of the solution, and Mourre estimates allow to prove that the solution exits any compact set after some time, and that it remains microlocalized far from $\xi_1 = 0$.

3.2.1. *Short time behaviour.* The aim of this paragraph is to prove the following result. It shows that the solutions of (3.1) exit the support of $\bar{u}$ after a time $t_{\rm exit}\varepsilon$, for $|t_{\rm exit}|$ large enough (independent of ε). We only state the forward in time result: the backwards result is identical, up to changing the sign of time. We shall further restrict the analysis to T_+ since the argument for T_- is identical, up to some sign changes.

Proposition 3.2. *Let φ^0 be a function, microlocalized in a compact set $\mathcal{C}$ satisfying Assumption* (1.7) *and let φ be the associate solution of* (3.1)*. Let $[u_-, u_+]$ be a closed interval of $\mathbf{R}$ containing $Supp_{x_1}\bar{u}$. There exists a constant $t_{\rm exit} > 0$ such that for any $\varepsilon \in]0,1[$, the function $\varphi(\varepsilon t_{\rm exit}, \cdot)$ is microlocalized in a compact set $\mathcal{K}$ such that the projection of $\mathcal{K}$ onto the x_1-axis does not interesect $[u_-, u_+]$. Moreover $\mu Supp_{\xi_1}\varphi$ is unchanged. More precisely, if $\mu Supp_{\xi_1}\varphi^0 \subset \mathbf{R}^+ \setminus \{0\}$, then $\mu Supp_{x_1}\varphi(\varepsilon t_{\rm exit}, \cdot) \subset]u_+, +\infty[$, and if $\mu Supp_{\xi_1}\varphi^0 \subset \mathbf{R}^- \setminus \{0\}$, then $\mu Supp_{x_1}\varphi(\varepsilon t_{\rm exit}, \cdot) \subset] - \infty, u_-[$.*

Proof. Define the function $\psi(s) := \varphi(\varepsilon s)$. Then (3.1) reads

$$i\varepsilon\partial_s\psi = T_+\psi, \quad \psi_{|s=0} = \varphi^0\,, \tag{3.2}$$

and any result proved on ψ on $[0, \mathcal{T}]$ will yield the same result for φ on $[0, \mathcal{T}\varepsilon]$. Notice that (3.2) is written in a semi-classical setting, so by the propagation of the microsupport theorem, the microsupport of ψ is propagated by the bicharacteristics, which are the integral curves of the principal symbol. The bicharacteristics are given by the following set of ODEs:

$$\begin{cases} \dot{x}^t = \nabla_\xi\tau_+(\xi_1^t, x_2^t, \xi_2^t), & x^0 = (x_1^0, x_2^0) \\ \dot{\xi}^t = -\nabla_x\tau_+(\xi_1^t, x_2^t, \xi_2^t), & \xi^0 = (\xi_1^0, \xi_2^0)\,. \end{cases}$$

Notice that τ_+ is independent of x_1, so $\dot{\xi}_1^t$ is identically zero and therefore $\xi_1^t \equiv \xi_1^0$. So for all $s \geq 0$, the microlocal support in ξ_1 of $\psi(s)$ remains unchanged, and in particular is far from $\xi_1 = 0$. Moreover one has

$$\dot{x}_1^t = \frac{\xi_1^0}{\sqrt{(\xi_1^0)^2 + (\xi_2^t)^2 + b^2(x_2^t)}}\,\cdot$$

Now we recall that the bicharacteristic curves lie on energy surfaces, meaning that on each bicharacteristic, $\tau_+(\xi_1^0, x_2^t, \xi_2^t)$ is a constant. That implies that $(\xi_2^t)^2 + b^2(x_2^t)$ is a constant on each bicharacteristic, so that for all times,

$$\dot{x}_1^t \equiv \frac{\xi_1^0}{\sqrt{(\xi_1^0)^2 + (\xi_2^0)^2 + b^2(x_2^0)}}\,\cdot$$

If $\xi_1^0 > 0$, then x_1 is propagated to the right and eventually escapes to the right of the support in x_1 of $\bar{u}$, whereas if $\xi_1^0 < 0$, the converse (to the left) occurs. Proposition 3.2 is proved. □

3.2.2. *Long time behaviour.* The aim of this paragraph is to prove the following result, which again is only proved for positive times for simplicity.

Proposition 3.3. *Under the assumptions of Proposition* 3.2*, let φ^+ be the solution of* (3.1) *associated with the data $\varphi(\varepsilon t_{\rm exit}, \cdot)$. Then $\mu Supp_{x_1}\varphi^+(t)$ does not intersect $Supp_{x_1}\bar{u}$ for $t \geq \varepsilon t_{\rm exit}$, and $\mu Supp_{\xi_1}\varphi^+(t)$ remains unchanged for $t \geq \varepsilon t_{\rm exit}$. Finally $\mu Supp_{x_1}\varphi^+(t)$ exits any compact set in x_1 as soon as $t > \varepsilon t_{\rm exit}$.*

Proof. Before going into the proof, we shall simplify the analysis by only studying the case of T_+ (the case T_- is obtained by identical arguments), and we shall only deal with the case when the support in ξ_1 of the data lies in the positive half space. The other case is obtained similarly.

The proof is based on Mourre's theory which we shall now briefly recall, and we refer to [13] and [12] for all details. Let us consider two self-adjoint operators H and A on a Hilbert space $\mathcal{H}$. We make the following assumptions:

(1) the intersection of the domains of A and H is dense in the domain $\mathcal{D}(H)$ of H .
(2) $t \mapsto e^{itA}$ maps $\mathcal{D}(H)$ to itself, and for all $\varphi^0 \in \mathcal{D}(H)$,
$$\sup_{t\in[0,1]} \|He^{itA}\varphi^0\| < \infty\,.$$
(3) The operator $i[H,A]$ is bounded from below and closable, and the domain $\mathcal{D}(B_1)$ where iB_1 is its closure, contains $\mathcal{D}(H)$. More generally for all $n \in \mathbf{N}$ the operator $i[iB_n, A]$ is bounded from below and closable and the domain $\mathcal{D}(B_{n+1})$ of its closure iB_{n+1} contains $\mathcal{D}(H)$, and finally B_{n+1} extends to a bounded operator from $\mathcal{D}(H)$ to its dual.
(4) There exists $\theta > 0$ and an open interval Δ of $\mathbf{R}$ such that if E_Δ is the corresponding spectral projection of H, then

$$E_\Delta i[H,A]E_\Delta \geq \theta E_\Delta\,. \tag{3.3}$$

Note that Assumptions (1 - 3) can be replaced by the fact that $[f(H),A]$ and all commutator iterates are bounded for any smooth, compactly supported function f (see [12]).

Under those assumptions, for any integer $m \in \mathbf{N}$ and for any $\theta' \in]0,\theta[$, there is a constant C such that
$$\|\chi_-(A-a-\theta' t)e^{-iHt}g(H)\chi_+(A-a)\| \leq Ct^{-m}$$
where $\chi_\pm$ is the characteristic function of $\mathbf{R}^\pm$, g is any smooth compactly supported function in Δ, and the above bound is uniform in $a \in \mathbf{R}$.

Let us apply this theory to our situation. We consider equation (3.1) with data $e^{i\frac{t_{\rm exit}}{\varepsilon}T_+}\varphi_+^0$, and let us define the operator T_+^0 as the operator T_+ where $\bar u$ has been chosen identically zero. We shall start by studying the equation

$$i\varepsilon^2\partial_t\widetilde\varphi = T_+^0\widetilde\varphi, \quad \widetilde\varphi_{|t=\varepsilon t_{\rm exit}} = e^{i\frac{t_{\rm exit}}{\varepsilon}T_+}\varphi_+^0\,, \tag{3.4}$$

for which we shall prove Proposition 3.3. Then we shall prove that the solution $\widetilde\varphi$ actually solves the original equation (3.1) with the same data $e^{i\frac{t_{\rm exit}}{\varepsilon}T_+}\varphi_+^0$ at $t = \varepsilon t_{\rm exit}$ up to $O(\varepsilon^\infty)$, because its support in x_1 lies outside the support of $\bar u$ and because the symbolic expansion of T_+ depends on x_1 only through $\bar u$ and its derivatives (see Remark 2.3).

So let us start by applying Mourre's theory to (3.4). Let us write the projection of $\mathcal{K}$ onto the ξ_1-axis as included in $[d_0, d_1]$ with $0 < d_0 < d_1 < \infty$. We recall that on the support of $e^{i\frac{t_{\rm exit}}{\varepsilon}T_+}\varphi_+^0$, x_1 remains to the right of the support of $\bar u$. Then we apply the theory to $H = T_+^0$ and to $A = x_1$ (the pointwise multiplication). Assumptions (1) to (3) are easy to check, in particular because this is a semiclassical setting, so only the principal symbols need to be considered. Similarly finding a lower bound for $E_\Delta i[T_+^0, x_1]E_\Delta$ boils down to computing the Poisson bracket $\{\tau_+, x_1\}$ and one finds

$$\{\tau_+, x_1\} = \frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}} \cdot \tag{3.5}$$

Since T_+^0 has constant coefficients in x_1, ξ_1 is preserved, so in particular for all times one has $\mu\mathrm{Supp}_{\xi_1}\widetilde{\varphi}(t) \subset [d_0, d_1]$. One can furthermore choose for Δ an interval of $\mathbf{R}$ of the type $]D_0, D_1[$ where the constants D_0 and D_1 are chosen so that for any $(x,\xi) \in \mathcal{K}$, one has

$$D_0 < \sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)} < D_1 \,. \tag{3.6}$$

As the microlocal supports of the eigenfunctions of T_+^0 lie on energy surfaces, we know that the solution to (3.4) will remain in E_Δ for all times. Now let us apply the results of [13] and [12]. By (3.5), (3.6) and the assumption on ξ_1 written above, we have that

$$E_\Delta i[H,A]E_\Delta \geq \varepsilon \frac{d_0}{D_1} E_\Delta \,,$$

so (3.3) holds with $\theta = \varepsilon d_0/D_1$. It follows that the solution $e^{i\frac{(t-\varepsilon t_{\mathrm{exit}})}{\varepsilon^2}T_+^0}\big(e^{i\frac{t_{\mathrm{exit}}}{\varepsilon}T_+}\varphi_+^0\big)$ to (3.4) has a support in x_1 such that

$$x_1 > u_+ + \frac{d_0}{D_1}\frac{t}{\varepsilon}$$

which proves the result for (3.4).

Since $\mu\mathrm{Supp}_{x_1} e^{i\frac{(t-\varepsilon t_{\mathrm{exit}})}{\varepsilon^2}T_+^0}\big(e^{i\frac{t_{\mathrm{exit}}}{\varepsilon}T_+}\varphi_+^0\big)$ does not cross $\mathrm{Supp}_{x_1}\bar{u}$, one has actually

$$e^{i\frac{(t-\varepsilon t_{\mathrm{exit}})}{\varepsilon^2}T_+^0}\big(e^{i\frac{t_{\mathrm{exit}}}{\varepsilon}T_+}\varphi_+^0\big) = e^{i\frac{(t-\varepsilon t_{\mathrm{exit}})}{\varepsilon^2}T_+}\big(e^{i\frac{t_{\mathrm{exit}}}{\varepsilon}T_+}\varphi_+^0\big) \quad \text{in } L^2$$

locally uniformly in t, due to the following lemma. The proposition follows. □

Lemma 3.4. *Let A_ε and $\tilde{A}_\varepsilon$ be two pseudo-differential operators such that*

- *iA_ε is hermitian in $L^2(\mathbf{R}^d)$,*
- *$A_\varepsilon - \tilde{A}_\varepsilon = O(\varepsilon^\infty)$ microlocally on $\Omega \subset \mathbf{R}^{2d}$.*

Let $\tilde{\varphi}$ be a solution to $i\partial_t\tilde{\varphi} + \tilde{A}_\varepsilon\tilde{\varphi} = 0$ microlocalized in Ω, and φ the solution to $i\partial_t\varphi + A_\varepsilon\varphi = 0$ with the same initial data. Then, for all $N \in \mathbf{N}$,

$$\sup_{t \leq \varepsilon^{-N}} \|\varphi(t) - \tilde{\varphi}(t)\|_{L^2(\mathbf{R}^d)} = O(\varepsilon^\infty) \,.$$

Proof. The proof is based on a simple energy inequality and is completely straightforward. We have

$$\begin{aligned}
\frac{d}{dt}\|\varphi - \tilde{\varphi}\|^2_{L^2(\mathbf{R}^d)} &= 2\langle iA_\varepsilon\varphi - i\tilde{A}_\varepsilon\tilde{\varphi}|\varphi - \tilde{\varphi}\rangle \\
&= 2\langle (iA_\varepsilon - i\tilde{A}_\varepsilon)\tilde{\varphi}|\varphi - \tilde{\varphi}\rangle \\
&\leq 2\|(A_\varepsilon - \tilde{A}_\varepsilon)\tilde{\varphi}\|_{L^2(\mathbf{R}^d)}\|\varphi - \tilde{\varphi}\|_{L^2(\mathbf{R}^d)} \,.
\end{aligned}$$

This leads to

$$\|\varphi(t) - \tilde{\varphi}(t)\|^2_{L^2(\mathbf{R}^d)} = O(\varepsilon^\infty)t \,,$$

which concludes the proof. □

3.3. **An alternate proof.** In the case when b^2 has at most one, non degenerate critical point, one can use a more direct, semi-explicit method to obtain the result. The first step consists in taking the Fourier transform in x_1 (recalling that the problem is invariant by translations in the x_1 direction). We also recall that the data is microlocalized on a compact set $\mathcal{C}$ such that ξ_1 is bounded away from zero. Since $\tau_\pm = \pm\sqrt{\xi_2^2 + b^2(x_2) + \xi_1^2}$, functional calculus implies that one can find classical pseudo-differential operators $H_\pm(\xi_1)$ of principal symbols $\xi_2^2 + b^2(x_2)$ such that $T_\pm = \pm\sqrt{H_\pm(\xi_1) + \xi_1^2}$. Let us now call $\lambda_\pm^k(\xi_1)$ and $\varphi_\pm^k(\xi_1; x_2)$ the eigenvalues and eigenfunctions of $H_\pm(\xi_1)$. We refer to [5] for a proof of the following proposition.

Proposition 3.5. *Let ϕ be an eigenfunction of $H_\pm(\xi_1)$, microlocalized on an energy surface which interstects $\mathcal{C}$. Then ϕ and its associate eigenvalue λ are C^∞ functions of ξ_1. Moreover $\frac{1}{\varepsilon}\partial_{\xi_1}\lambda$ is bounded on compact sets in ξ_1.*

Let us now carry out this program. We consider an initial data denoted φ^0, microlocalized in $\mathcal{C}$. One can take the Fourier transform in x_1 which gives

$$\varphi^0(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \int \hat{\varphi}^0(\xi_1, x_2) e^{-i\frac{x_1\xi_1}{\varepsilon}}\, d\xi_1\,.$$

Now let us consider a coherent state (in Fourier variables) at (q, p) (see Appendix A), that is:

$$\varphi_{qp}(\xi_1) := \frac{1}{(\pi\varepsilon)^{\frac{1}{4}}} e^{i\frac{\xi_1 q}{\varepsilon}} e^{-\frac{(\xi_1 - p)^2}{2\varepsilon}}\,. \tag{3.7}$$

After decomposition onto coherent states we get

$$\varphi^0(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \frac{1}{(\pi\varepsilon)^{\frac{1}{4}}} \int \tilde{\varphi}^0(q, p, x_2) e^{i\frac{\xi_1(q - x_1)}{\varepsilon}} e^{-\frac{(\xi_1 - p)^2}{2\varepsilon}}\, dq dp d\xi_1$$

where $\tilde{\varphi}^0(q, p, x_2) := \big(\varphi_{qp} | \hat{\varphi}^0(\cdot, x_2)\big)_{L^2}$. We notice that the integral over p and q is, modulo $O(\varepsilon^\infty)$, on a compact domain due to the microlocalization assumption on φ^0. Finally decomposing onto the eigenfunctions $\varphi_\pm^k(\xi_1; x_2)$ gives

$$\varphi^0(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \frac{1}{(\pi\varepsilon)^{\frac{1}{4}}} \sum_k \int \overline{\varphi}^0(q, p; k, \xi_1) \varphi_\pm^k(\xi_1; x_2) e^{i\frac{\xi_1(q - x_1)}{\varepsilon}} e^{-\frac{(\xi_1 - p)^2}{2\varepsilon}}\, dq dp d\xi_1$$

where $\overline{\varphi}^0(q, p; k, \xi_1) := \Big(\varphi_\pm^k(\xi_1; \cdot) | \tilde{\varphi}^0(q, p, \cdot)\Big)_{L^2}$. Note that the dependence of $\overline{\varphi}^0$ on ξ_1 is only through the eigenfunction $\varphi_\pm^k$, so $\overline{\varphi}^0$ depends smoothly on ξ_1, as stated in Proposition 3.5.

The sum over k contains $O(\varepsilon^{-1})$ terms, due to the fact that $\lambda_\pm^k(\xi_1)$ remains in a finite interval (this can be seen in the proof of Proposition 3.5 in [5]). Now it remains to propagate at time t/ε^2 this initial data, which gives rise to the following expression:

$$\frac{1}{\sqrt{2\pi\varepsilon}} \frac{1}{(\pi\varepsilon)^{\frac{1}{4}}} \sum_k \int \overline{\varphi}^0(q, p; k, \xi_1) \varphi_\pm^k(\xi_1; x_2) e^{i\frac{\xi_1(q - x_1)}{\varepsilon}} e^{-\frac{(\xi_1 - p)^2}{2\varepsilon}} e^{\pm i(\lambda_\pm^k(\xi_1) + \xi_1^2)^{\frac{1}{2}} \frac{t}{\varepsilon^2}}\, dq dp d\xi_1\,.$$

The stationary phase lemma then gives that this integral is $O(\varepsilon^\infty)$ except if there exists a stationary point, given by the conditions:

$$\xi_1 = p \text{ and } \varepsilon(x_1 - q) \pm \frac{2\xi_1 + \partial_{\xi_1}\lambda_\pm^k(\xi_1)}{2\sqrt{\lambda_\pm^k(\xi_1) + \xi_1^2}} t = 0\,.$$

The second condition gives

$$2\sqrt{\lambda^k_\pm(\xi_1)+\xi_1^2}(x_1-q) = \mp\frac{1}{\varepsilon}\left(2p+\partial_{\xi_1}\lambda^k_\pm(\xi_1)\right)t\,.$$

Therefore, since $p\neq 0$ and the $\lambda^k_\pm$'s are bounded, with $\partial_{\xi_1}\lambda^k_\pm(\xi_1)=O(\varepsilon)$, there is no critical point for x_1 in a compact set. Proposition 3.5 therefore allows to apply the stationary phase lemma and to conclude the proof of Theorem 3. Notice that the (fixed) losses in ε (namely the negative powers of ε appearing in the integrals and the number of k's in the sum) are compensated by the fact that the result is $O(\varepsilon^\infty)$; it is important at this point that as noticed above, the function $\overline{\varphi}^0$ depends smoothly on ξ_1.

4. Trapping of Rossby waves

In this section we shall prove that Rossby waves are trapped if the initial data is correctly microlocalized. Recall that the Rossby projection $\mathbf{v}^R_\varepsilon$ of the solution $\mathbf{v}_\varepsilon$ to (1.9) satisfies the equation

$$\text{(4.1)}\qquad \begin{cases} i\varepsilon\partial_t\varphi = T_R\varphi \\ \varphi_{|t=0}=\varphi^0, \end{cases} \qquad \sigma_P(T_R)=\tau_R=\frac{\xi_1 b'}{\xi_1^2+\xi_2^2+b(x_2)^2}+\bar u\cdot\xi\,.$$

The first point one must check is that the solution to that equation remains microlocalized in a set satisfying the separation condition $\xi_1^2+\xi_2^2+b(x_2)^2\neq 0$. This is performed in the next Section 4.1.

In the case of a general $\bar u$ we are only able to prove trapping in the latitude direction. The result is the following.

Theorem 4 (Trapping in the latitude direction)**.** *Let $\mathbf{v}_{\varepsilon,0}$ be a family of initial data, microlocalized in a compact set $\mathcal{C}$ satisfying Assumption* (1.7)*. For any parameter $\varepsilon>0$, denote by $\mathbf{v}^R_\varepsilon$ the Rossby component of the solution $\mathbf{v}_\varepsilon$ to* (1.9) *constructed in Theorem* 1*. Then there is a compact set* K_2 *of* $\mathbf{R}$ *such that*

$$\forall t\geq 0,\quad \mu\mathrm{Supp}_{x_2}\mathbf{v}^R_\varepsilon(t)\subset \mathrm{K}_2\,.$$

In the case when $\bar u$ is a shear flow, $\bar u=(\bar u_1(x_2),0)$, the analysis can be made more precise. We assume in the following that there are two points $y_1\neq y_2$ in $\mathbf{R}$ such that $\bar u_1(y_1)=\bar u_1(y_2)=0$, and for instance that $\bar u_1<0$ on $]y_1,y_2[$. We also assume that b does not change sign on $]y_1,y_2[$. We prove the following result.

Theorem 5 (Trapping in the longitude direction, the shear flow case)**.** *Under the above assumptions on $\bar u$ and b, there is a subset Λ of codimension one and a compact set* C *of* $\mathbf{R}^2$ *such that any family $\mathbf{v}_{\varepsilon,0}$ of initial data microlocalized in Λ satisfies the following property: the Rossby component $\mathbf{v}^R_\varepsilon$ of the solution $\mathbf{v}_\varepsilon$ to* (1.9) *constructed in Theorem* 1 *satisfies*

$$\forall t>0,\quad \|\mathbf{v}^R_\varepsilon(t)\|_{L^2(\mathrm{C})}=O(1)\,.$$

The proof of Theorem 4 is given in Section 4.2 while Section 4.3 is devoted to the proof of Theorem 5.

4.1. **Microlocalization.** Because of the scaling of the Rossby hamiltonian, on the times scales considered here the propagation of energy by Rossby waves is described by the hamiltonian dynamics

$$\frac{dx_i}{dt} = \frac{\partial \tau_R}{\partial \xi_i}, \quad \frac{d\xi_i}{dt} = -\frac{\partial \tau_R}{\partial x_i}$$

which can be written explicitly

$$\begin{aligned}
\frac{dx_1}{dt} &= b'(x_2)\frac{\langle \xi \rangle_b^2 - 2\xi_1^2}{\langle \xi \rangle_b^4} + \bar{u}_1(x),\\
\frac{dx_2}{dt} &= -2b'(x_2)\frac{\xi_1 \xi_2}{\langle \xi \rangle_b^4} + \bar{u}_2(x),\\
\frac{d\xi_1}{dt} &= -\partial_1 \bar{u}_1(x)\xi_1 - \partial_1 \bar{u}_2(x)\xi_2,\\
\frac{d\xi_2}{dt} &= \xi_1 \frac{2b(b')^2 - b''\langle \xi \rangle_b^2}{\langle \xi \rangle_b^4} - \partial_2 \bar{u}_1(x)\xi_1 - \partial_2 \bar{u}_2(x)\xi_2
\end{aligned} \tag{4.2}$$

where we recall that $\langle \xi \rangle_b = \sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$. In order for the dynamics to be well defined and also in order to justify the diagonalization process, we need the quantity $\langle \xi \rangle_b$ to remain bounded from below for all times.

Proposition 4.1. *Let $\mathcal{C}$ be some compact subset of $\mathbf{R}^4$ such that*

$$\mathcal{C} \cap \{(x_1, x_2, \xi_1, \xi_2) \,/\, \xi_1^2 + \xi_2^2 + b^2(x_2) = 0\} = \emptyset\,.$$

Then the bicharacteristics $t \mapsto (x(t), \xi(t))$ of the Rossby Hamiltonian starting from any point $(x_1^0, x_2^0, \xi_1^0, \xi_2^0)$ of $\mathcal{C}$ are defined globally in time, and $\forall t \in \mathbf{R}$,

$$\inf_{(x_1^0, x_2^0, \xi_1^0, \xi_2^0) \in \mathcal{C}} (\xi_1(t)^2 + \xi_2(t)^2 + b^2(x_2(t))) > 0.$$

Proof. As b', b'', u and Du are Lipschitz, by the Cauchy-Lipschitz theorem the system of ODEs (4.2) has a unique maximal solution. In order to prove that this solution is defined globally, it is enough to prove that the time derivative of this solution is uniformly bounded. This comes from assumption (1.6) giving an upper bound on $b'/\big(1 + b^2(x_2)\big)^{\frac{1}{2}}$ and $b''/\big(1 + b^2(x_2)\big)^{\frac{1}{2}}$, and from the lower bound on $\langle \xi \rangle_b$ to be established now.

The crucial assumption here is the fact that b' and b do not vanish simultaneously.

So let us suppose that $\langle \xi \rangle_b$ vanishes, and consider the first time t^* at which $\langle \xi \rangle_b(t^*) = 0$. Assume to start with that $x(t^*)$ lies outside the support of $\bar{u}$. Then there is a small amount of time (t^-, t^*), $t^- < t^*$, on which $x(t)$ remains outside the support of $\bar{u}$. So on the interval $(t^-, t^*), \xi_1$ is a constant hence remains zero, and an inspection of the ODEs then shows that on $(t^-, t^*), x_2$ and ξ_2 are also constant, hence $\langle \xi \rangle_b(t) = 0$ which is impossible by definition of t^*.

Now let us assume that $x(t^*)$ does not lie outside the support of $\bar{u}$, where t^* is still the first time t^* at which $\langle \xi \rangle_b(t^*) = 0$, assuming such a time exists. We shall prove that

$$|\langle \xi \rangle_b(t)| \lesssim (t^* - t)^{\frac{1}{2}}, \quad t \to t^*. \tag{4.3}$$

Indeed we have clearly

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle\xi\rangle_b^2 &= b(x_2)b'(x_2)\frac{dx_2}{dt}+\xi_1\frac{d\xi_1}{dt}+\xi_2\frac{d\xi_2}{dt}\\
&= b(x_2)b'(x_2)\bar u_2(x)-\partial_1\bar u(x)\cdot\xi-\partial_2\bar u(x)\cdot\xi+\frac{b''(x_2)\xi_1\xi_2}{\langle\xi\rangle_b^2}
\end{aligned}
\tag{4.4}
$$

so in particular we find that $\frac{d}{dt}\langle\xi\rangle_b^2$ is bounded as t goes to t^*, hence (4.3) holds.

Moreover along a trajectory of the Rossby Hamiltonian, τ_R is conserved, and we have

$$
\frac{dx_1}{dt}=\frac{b'(x_2)}{\langle\xi\rangle_b^2}-\frac{2(\tau_R-u(x)\cdot\xi)^2}{b'(x_2)}+\bar u_1(x).
$$

Since b' and b do not vanish simultaneously, this in turn implies that there is a constant C such that as t goes to t^*,

$$
\Big|\frac{dx_1}{dt}\Big|\geq\frac{C}{t^*-t}.
$$

In particular there is a time $t<t^*$ at which the trajectory has escaped the support of $\bar u$, which is contrary to our assumption. This concludes the proof of the proposition. □

4.2. **Trapping in the latitude direction.**

Proof of Theorem 4. The energy surfaces corresponding to $\tau_R\neq 0$ are bounded in the x_2 direction : as $x_2\to\pm\infty$,

$$
\frac{b'(x_2)\xi_1}{\langle\xi\rangle_b^2}+\bar u(x)\cdot\xi\to 0\,.
$$

Consider now a trajectory on the energy level $\tau_R=0$, and some point of this trajectory (y_1,y_2,ξ_1,ξ_2) such that $y_2\notin\mathrm{Supp}_{x_2}\bar u$. One has

$$
b'(y_2)\xi_1=0\,.
$$

- If $b'(y_2)=0$, then

$$
\frac{dx_1}{dt}=\frac{dx_2}{dt}=\frac{d\xi_1}{dt}=\frac{d\xi_2}{dt}=0\,.
$$

The uniqueness in Cauchy-Lipschitz theorem implies then that the trajectory is nothing else than a fixed point, and therefore in particular is bounded.

- If $\xi_1=0$, then

$$
\frac{dx_2}{dt}=\frac{d\xi_1}{dt}=\frac{d\xi_2}{dt}=0 \quad\text{and}\quad \frac{dx_1}{dt}=b'(y_2)\frac{\xi_2^2+b^2(y_2)-\xi_1^2}{\langle\xi\rangle_b^4}\,,
$$

meaning that the trajectory is a uniform straight motion along x_1. In particular, it is bounded in the x_2-direction.

Finally, we conclude that trajectories on the energy level $\tau_R=0$ are either trapped in the support $\mathrm{Supp}_{x_2}\bar u$, or trivial in the x_2-direction. □

4.3. **Trapping in the longitude direction in the shear flow case.**

Proof of Theorem 5. The strategy to study the qualitative behaviour of the trajectories is to first (in Paragraph 4.3.1) consider the motion in the reduced phase space $(x_2, \xi_2) \in \mathbf{R}^2$ and then (in Paragraph 4.3.2) to study the motion in the x_1 direction.

4.3.1. *Trajectories in the reduced* (x_2, ξ_2) *phase space.* In this section we study the trajectories in the reduced (x_2, ξ_2) phase space. We shall denote $\xi_1 := \xi_1^0$.

Since the Hamiltonian $\widetilde{\tau}_R$ and ξ_1 are conserved along any trajectory, trajectories are submanifolds of

$$\mathcal{E}_{\tau,\xi_1} := \big\{(x_2, \xi_2) \in \mathbf{R}^2 \,;\, \widetilde{\tau}_R(\xi_1, x_2, \xi_2) = \tau\big\}.$$

In the following we shall note for any energy τ and any $\xi_1 \in \mathbf{R}^*$

$$V_{\tau,\xi_1}(x_2) := \frac{b'(x_2)\xi_1}{\tau - \bar{u}_1(x_2)\xi_1} - \xi_1^2 - b^2(x_2),$$

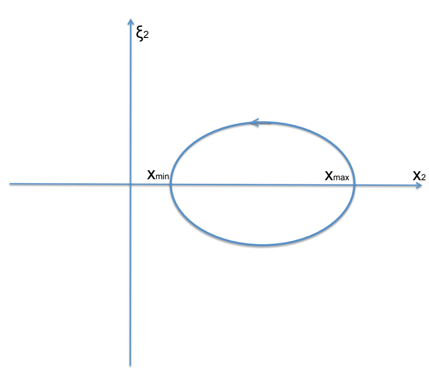
so that if $\mathcal{D} := \big\{x_2 / V_{\tau,\xi_1}(x_2) \geq 0\big\}$, then $\mathcal{E}_{\tau,\xi_1} = \Big\{\big(x_2, \pm\sqrt{V_{\tau,\xi_1}(x_2)}\,\big),\ x_2 \in \mathcal{D}\Big\}$. Note that $V_{\tau,\xi_1}(x_2^t)$ becomes singular if x_2^t reaches a point x_2 such that $\tau = \bar{u}_1(x_2)\xi_1$.

Several types of trajectories can correspond to such a system (see [5]), we shall isolate two particular types of trajectories here: periodic ones, and asymptotic ones.

Periodic trajectories: These correspond to the case when there exists $[x_{min}, x_{max}]$ in $\mathbf{R}$ with $x_{min} \neq x_{max}$, containing x_2^0 such that

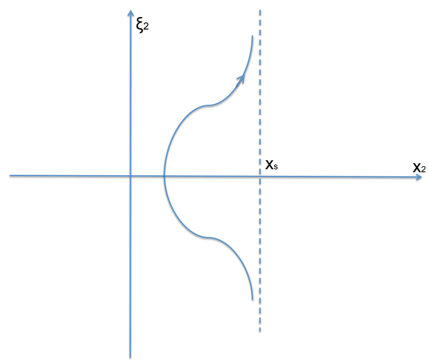
- V_{τ,ξ_1} has no singularity and does not vanish on $]x_{min}, x_{max}[$;
- $V_{\tau,\xi_1}(x_{min}) = V_{\tau,\xi_1}(x_{max}) = 0$;
- the points x_{min} and x_{max} are reached in finite time.

The extremal points x_{min} and x_{max} are then turning points, meaning that the motion is periodic, with a typical phase portrait as shown in the figure.



Asymptotic trajectories: These correspond to the case when there exists an interval $[x_{min}, x_{max}]$ of $\mathbf{R}$ containing x_2^0 such that

- V_{τ,ξ_1} has no singularity and does not vanish on $]x_{min}, x_{max}[$;
- x_{min} and x_{max} are either zeros or singularities of V_{τ,ξ_1};
- as $t \to \infty$, $x_2^t \to x_2^\infty$ where $x_2^\infty \in \{x_{min}, x_{max}\}$ is a pole of multiplicity 1 of V_{τ,ξ_1}. For the sake of simplicity, we further impose that $b'(x_2^\infty) \neq 0$.



Let us compute the rate of convergence of ξ_2 to infinity: without loss of generality, we can consider the case when the asymptotic point is x_{max}. Then we recall that

$$\lim_{t\to+\infty} \xi_2^t = \infty \quad \text{and} \quad \lim_{t\to+\infty} x_2^t = x_2^\infty, \quad \text{with} \quad \xi_1 \bar u_1(x_2^\infty) = \tau.$$

As x tends to x_2^∞, we have (recalling that $b'(x_2^\infty) \neq 0$)

$$V_\tau(x) \sim -\frac{b'(x_2^\infty)}{\bar u_1'(x_2^\infty)}(x - x_2^\infty)^{-1}.$$

This implies that

$$|\xi_2^t|^2 \sim -\frac{b'(x_2^\infty)}{\bar u_1'(x_2^\infty)}(x - x_2^\infty)^{-1} \quad \text{and} \quad \dot x_2^t \sim -2b'(x_2^\infty)\xi_1 \frac{\xi_2^t}{|\xi_2^t|^4} \sim 2\frac{|\bar u_1'(x_2^\infty)|^{3/2}|\xi_1|}{|b'(x_2^\infty|^{1/2}}(x_2^\infty - x)^{3/2}.$$

By integration, we get

$$x_2^t \sim x_2^\infty + C_1\, t^{-2}, \qquad \xi_2^t \sim C_2 t.$$

4.3.2. *Analysis of the trajectories in the x_1 direction: trapping phenomenon.* Let us prove the following result.

Proposition 4.2. *A necessary and sufficient condition for a periodic or an asymptotic trajectory to be trapped is*

$$\lim_{t\to T} \frac{1}{t}\int_0^t \dot x_1^s ds = 0,$$

where T denotes the (finite) period of the motion along x_2 in the periodic case, and $T = +\infty$ in the asymptotic case.

Proof. We will study separately the different situations described in the previous section, namely the case of periodic and asymptotic trajectories in (x_2, ξ_2).

• In the case of a periodic motion in (x_2, ξ_2) of period $T > 0$, the function $\dot{x}_1^t$ is also periodic, with the same period. Writing

$$x_1^t = x_1^0 + \int_0^t \left(\dot{x}_1^s - \frac{1}{T} \int_0^T \dot{x}_1^{s'} \, ds' \right) ds + \frac{t}{T} \int_0^T \dot{x}_1^s \, ds$$

we see that depending on the average of $\dot{x}_1^t$ over $[0, T]$, x_1^t is either a periodic function, or the sum of a periodic function and a linear function. It follows that trapped trajectories are characterized by the criterion $\displaystyle\int_0^T \dot{x}_1^t dt = 0$.

• For asymptotic motions, we need to check that

$$\dot{x}_1^t - \bar{u}_1(x_2^\infty) t \;\text{ is integrable at infinity.}$$

We have indeed

$$\dot{x}_1^t = \bar{u}_1(x_2^t) + \frac{b'(x_2^t)(-\xi_1^2 + {\xi_2^t}^2 + b^2(x_2^t))}{(\xi_1^2 + {\xi_2^t}^2 + b^2(x_2^t))^2},$$

which, together with the asymptotic expansions of x_2^t and ξ_2^t obtained in the previous section, implies that

$$\dot{x}_1^t = \bar{u}_1(x_2^\infty) + O(t^{-2}).$$

It is then clear that the trajectory is trapped if and only if

$$\bar{u}_1(x_2^\infty) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \dot{x}_1^s ds = 0\,.$$

That proves the proposition. □

Now we shall conclude the proof of Theorem 5 by constructing a subset of $\mathbf{R}^4$ of codimension one in which all initial data lead to a trapped asymptotic trajectory. As we want to study trapped asymptotic trajectories, we restrict our attention to the case when $\tau = 0$, which is a necessary condition for asymptotic trajectories to be trapped. Extremal points of the trajectories are then defined in terms of the function

$$V_{0,\xi_1}(y) = -\frac{b'(y)}{\bar{u}_1(y)} - b(y)^2 - \xi_1^2\,.$$

Let us define

$$\varrho(y) := -\frac{b'(y)}{\bar{u}_1(y)} - b(y)^2\,, \qquad y \in]y_1, y_2[\,.$$

By definition of y_1 and y_2 (see the introduction of this part), one has

$$\lim_{y \to y_1+} \varrho(y) = +\infty \quad \text{and} \quad \lim_{y \to y_2-} \varrho(y) = +\infty.$$

Let us define $N := \max\big(0\,; \inf_{y \in]y_1, y_2[} \varrho(y)\big) \in \mathbf{R}_+$. For ξ_1 such that $\xi_1^2 \geq N$, we then define

$$h(\xi_1) := \sup \left\{ y \in]-\infty, y_2[\,;\; \varrho(y) \leq \xi_1^2 \right\} \in]y_1, y_2[\,.$$

We therefore have that

$$\forall y \in]h(\xi_1), y_2[, \quad y \text{ is neither a turning point nor a singular point.}$$

As h is a decreasing function on $]-\infty, -\sqrt{N}]$, all $\xi_1 \in]-\infty, -\sqrt{N}]$ except a countable number are continuity points. Choose then some $\tilde{\xi}_1$ to be a continuity point of h and $\tilde{x}_2^0 \in]h(\tilde{\xi}_1), y_2[$. By continuity of h, there exists a neighborhood $\tilde{V}$ of $(\tilde{x}_2^0, \tilde{\xi}_1)$ such that

$$\forall (x_2^0, \xi_1) \in \tilde{V}, x_2^0 - h(\xi_1) > 0\,.$$

The set $\left\{\left(x_1^0, \xi_1, x_2^0, (\varrho(x_2^0) - \xi_1^2)^{\frac{1}{2}}\right);\ (x_1^0, \xi_1, x_2^0) \in \mathbf{R} \times \tilde{V}\right\}$ is a submanifold of $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^2$ having codimension 1. Furthermore, for any initial data in this set, we have $\dot{x}_{2|t=0} > 0$ and a simple connexity argument shows that x_2^t is an increasing function of time. In particular we have $x_2^t \to y_2$ as $t \to \infty$. This concludes the proof of Theorem 5. □

5. The nonlinear equation

In this final section we prove that the life span of the nonlinear equation may be bounded from below uniformly in ε. Moreover we check that the solution to the nonlinear equation remains close, in L^2, to the solution of the linear equation. The result is the following, which deals more generally with the next system, for $\eta \geq 0$:

$$\varepsilon^2 \partial_t U + A(x, \varepsilon D_x) U + \varepsilon^{3+\eta} S_1(U) \varepsilon \partial_1 U + \varepsilon^{3+\eta} S_2(U) \varepsilon \partial_2 U = 0, \quad \eta \geq 0\,. \tag{5.1}$$

The case $\eta = 0$ corresponds of course to the original system (1.3) presented in the introduction.

Theorem 6 (The nonlinear equation)**.** *Let $U_{\varepsilon,0}$ be any initial data bounded in W_ε^4. Then the following results hold.*

(1) *The case $\eta = 0$:*
   (a) *There exists some $T^* > 0$ such that the initial value problem* (5.1) *with $\eta = 0$ has a unique solution U_ε on $[0, T^*[$ for any $\varepsilon > 0$.*
   (b) *Assume that the solution V_ε to the linear equation* (1.9) *satisfies*

   $$\|\varepsilon V_\varepsilon\|_{L^2([0,T^*[;L^\infty)} \to 0 \text{ as } \varepsilon \to 0.$$

   Then the solution U_ε to (5.1) *with $\eta = 0$ satisfies*

   $$\|U_\varepsilon - V_\varepsilon\|_{L^2} \to 0 \text{ uniformly on } [0, T^*[\text{ as } \varepsilon \to 0.$$

(2) *The case $\eta > 0$: Le $T > 0$ be fixed. Then there is $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, the equation* (5.1) *has a unique solution U_ε on $[0, T]$. Moreover,*

   $$\|U_\varepsilon - V_\varepsilon\|_{L^2} \to 0 \text{ uniformly on } [0, T] \text{ as } \varepsilon \to 0.$$

Remark 5.1. *The refined L^∞ estimate on the linear solution required in result* (1b) *should be proved by using*

- *some generalized WKB expansion for the Rossby waves (on Fourier side near caustics), together with estimates on Airy functions (note that before the first caustic the WKB expansion gives immediately the required estimate);*
- *some stationary phase lemma to estimate the contribution of Poincaré waves.*

Depending on the degeneracy of stationary points, we should gain some small power of ε due to the oscillatory behaviour of the integrals.

Proof of Theorem 6. Extending a result by Dutrifoy, Majda and Schochet [7] obtained in the particular case when $b(x_2) = \beta x_2$, we shall prove that there is an operator of principal symbol $(\xi_1^2 + \xi_2^2 + b^2)\mathrm{Id}$ which "almost commutes" with $A(x, \varepsilon D_x)$ in the semiclassical regime. The first step, as in [7], is to perform the following orthogonal change of variable

$$\tilde{U} := \left(\frac{u_0 + u_1}{\sqrt{2}}, \frac{u_0 - u_1}{\sqrt{2}}, u_2 \right)$$

in order to produce the generalized creation and annihilation operators

$$L_\pm := \frac{1}{\sqrt{2}} \big(\varepsilon \partial_2 \mp b \big) .$$

The system can indeed be rewritten

$$\varepsilon^2 \partial_t \tilde{U} + \tilde{A}(x, \varepsilon D_x) \tilde{U} + \varepsilon^3 \tilde{S}_1(\tilde{U}) \varepsilon \partial_1 \tilde{U} + \varepsilon^3 \tilde{S}_2(\tilde{U}) \varepsilon \partial_2 \tilde{U} = 0$$

with

$$\tilde{A}(x, \varepsilon D_x) := \begin{pmatrix} \varepsilon \bar{u}_1 \varepsilon \partial_1 + \varepsilon \partial_1 & 0 & L_+ \\ 0 & \varepsilon \bar{u}_1 \varepsilon \partial_1 - \varepsilon \partial_1 & L_- \\ L_- & L_+ & \varepsilon \bar{u}_1 \varepsilon \partial_1 \end{pmatrix} + O(\varepsilon^2 \mathrm{Id}) ,$$

and

$$\tilde{S}_1(\tilde{U}) := \begin{pmatrix} \frac{3\tilde{u}_0 - \tilde{u}_1}{2\sqrt{2}} & 0 & 0 \\ 0 & \frac{\tilde{u}_0 - 3\tilde{u}_1}{2\sqrt{2}} & 0 \\ 0 & 0 & \frac{\tilde{u}_0 - \tilde{u}_1}{\sqrt{2}} \end{pmatrix} ,$$

$$\text{and} \quad \tilde{S}_2(\tilde{U}) := \begin{pmatrix} \tilde{u}_2 & 0 & \frac{\tilde{u}_0 + \tilde{u}_1}{4} \\ 0 & \tilde{u}_2 & \frac{\tilde{u}_0 + \tilde{u}_1}{4} \\ \frac{\tilde{u}_0 + \tilde{u}_1}{4} & \frac{\tilde{u}_0 + \tilde{u}_1}{4} & \tilde{u}_2 \end{pmatrix} .$$

Next, remarking that $[\varepsilon^2 \partial_2^2 - b^2, \varepsilon \partial_2 \pm b] = \pm 2\varepsilon b' (\varepsilon \partial_2 \pm b) \pm \varepsilon^2 b''$, we introduce the operator

$$D_\varepsilon := \begin{pmatrix} \varepsilon^2 \partial_2^2 - b^2 + 2\varepsilon b' & 0 & 0 \\ 0 & \varepsilon^2 \partial_2^2 - b^2 - 2\varepsilon b' & 0 \\ 0 & 0 & \varepsilon^2 \partial_2^2 - b^2 \end{pmatrix} .$$

We notice that D_ε is a scalar operator at leading order. Moreover one can compute the commutator $[\varepsilon^2 \partial_1^2 + D_\varepsilon, \tilde{A}(x, \varepsilon D_x)]$: we find

$$[\varepsilon^2 \partial_1^2 + D_\varepsilon, \tilde{A}(x, \varepsilon D_x)] = O(\varepsilon^2 (\mathrm{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon)) \tag{5.2}$$

meaning that the commutator $[\varepsilon^2 \partial_1^2 + D_\varepsilon, \tilde{A}(x, \varepsilon D_x)]$ is of order $O(\varepsilon^2)$ with respect to the elliptic operator $\mathrm{Id} - \varepsilon^2 \partial_1^2 - D_\varepsilon$. That implies that the regularity of the solution to the linear equation

$$\varepsilon^2 \partial_t V + \tilde{A}(x, \varepsilon D_x) V = 0$$

can be controlled by an application of Gronwall's lemma: one has

$$\varepsilon^2 \|(Id - \varepsilon^2 \partial_1^2 - D_\varepsilon) V(t)\|_{L^2}^2 \le \varepsilon^2 \|(Id - \varepsilon^2 \partial_1^2 - D_\varepsilon) V_0\|_{L^2}^2 + C\varepsilon^2 \int_0^t \|(Id - \varepsilon^2 \partial_1^2 - D_\varepsilon) V(s)\|_{L^2}^2 ds ,$$

where C depends on the $W^{2,\infty}$ norms of $\bar{u}_1$ and b, so

$$\|(Id-\varepsilon^2\partial_1^2-D_\varepsilon)V(t)\|_{L^2}^2 \leq C\|(Id-\varepsilon^2\partial_1^2-D_\varepsilon)V_0\|_{L^2}^2\, e^{Ct}.$$

Now let us consider the nonlinear equation. Since the extended harmonic oscillator controls two derivatives in x_2, we get a control on the Lipschitz norm of U of the type

$$\|\varepsilon\partial_j U\|_{L^\infty} \leq \frac{C}{\varepsilon}(\|D_\varepsilon^2 U\|_{L^2}+\|\varepsilon^4\partial_1^4 U\|_{L^2}+\|U\|_{L^2})\,. \tag{5.3}$$

As D_ε is a scalar differential operator at leading order in ε, the antisymmetry of the higher order nonlinear term is preserved. More precisely, we have, using the Leibniz formula,

$$\begin{aligned}\varepsilon^2\partial_t D_\varepsilon\tilde{U}+\tilde{A}(x,\varepsilon D_x)D_\varepsilon\tilde{U}+\varepsilon^3\tilde{S}_1(\tilde{U})\varepsilon\partial_1 D_\varepsilon\tilde{U}+\varepsilon^3\tilde{S}_2(\tilde{U})\varepsilon\partial_2 D_\varepsilon\tilde{U}\\ =-[D_\varepsilon,\tilde{A}(x,\varepsilon D_x)]\tilde{U}-\varepsilon^3\tilde{S}_2(\tilde{U})[D_\varepsilon,\varepsilon\partial_2]\tilde{U}-\varepsilon^3[D_\varepsilon,\tilde{S}_j(\tilde{U})]\varepsilon\partial_j\tilde{U}\end{aligned}$$

as well as

$$\begin{aligned}\varepsilon^2\partial_t D_\varepsilon^2\tilde{U}+\tilde{A}(x,\varepsilon D_x)D_\varepsilon^2\tilde{U}+\varepsilon^3\tilde{S}_1(\tilde{U})\varepsilon\partial_1 D_\varepsilon^2\tilde{U}+\varepsilon^3\tilde{S}_2(\tilde{U})\varepsilon\partial_2 D_\varepsilon^2\tilde{U}=-[D_\varepsilon^2,\tilde{A}(x,\varepsilon D_x)]\tilde{U}\\ -\varepsilon^3\tilde{S}_2(\tilde{U})[D_\varepsilon,\varepsilon\partial_2]D_\varepsilon\tilde{U}-\varepsilon^3[D_\varepsilon,\tilde{S}_j(\tilde{U})]\varepsilon\partial_j D_\varepsilon\tilde{U}\\ +D_\varepsilon\left(-\varepsilon^3\tilde{S}_2(\tilde{U})[D_\varepsilon,\varepsilon\partial_2]\tilde{U}-\varepsilon^3[D_\varepsilon,\tilde{S}_j(\tilde{U})]\varepsilon\partial_j\tilde{U}\right).\end{aligned}$$

and in the same way, for $1\leq\ell\leq 4$,

$$\begin{aligned}\varepsilon^2\partial_t(\varepsilon\partial_1)^\ell\tilde{U}+\tilde{A}(x,\varepsilon D_x)(\varepsilon\partial_1)^\ell\tilde{U}+\varepsilon^3\tilde{S}_1(\tilde{U})(\varepsilon\partial_1)^{\ell+1}\tilde{U}+\varepsilon^3\tilde{S}_2(\tilde{U})\varepsilon\partial_2(\varepsilon\partial_1)^\ell\tilde{U}\\ =-\varepsilon^4\sum_{k=1}^{\ell}C_4^\ell(\varepsilon\partial_1)^\ell\tilde{S}_j(\tilde{U})\varepsilon\partial_j(\varepsilon\partial_1)^{\ell-k}\tilde{U}\,.\end{aligned}$$

In all cases, the terms of higher order disappear by integration in x and the other terms are controlled with the following trilinear estimate (writing generically $\tilde{Q}(\tilde{U})$ for all the nonlinearities): for all $0\leq k\leq 2$ and all $0\leq\ell\leq 4$

$$\begin{aligned}|<D_\varepsilon^k\tilde{U}|D_\varepsilon^k\tilde{Q}(\tilde{U})>|+|<(\varepsilon\partial_1)^\ell\tilde{U}|(\varepsilon\partial_1)^\ell\tilde{Q}(\tilde{U})>|\\ \leq C\|\tilde{U}\|_{W_\varepsilon^{1,\infty}}(\|D_\varepsilon^2\tilde{U}\|_{L^2}+\|(\varepsilon\partial_1)^4\tilde{U}\|_{L^2}+\|\tilde{U}\|_{L^2})^2\\ \leq\frac{C}{\varepsilon}\left(\|D_\varepsilon^2\tilde{U}\|_{L^2}+\|(\varepsilon\partial_1)^4\tilde{U}\|_{L^2}+\|\tilde{U}\|_{L^2}\right)^3.\end{aligned} \tag{5.4}$$

Remark 5.2. *Note that because of the bad embedding inequality $\|\nabla U\|_{L^\infty}\leq\dfrac{1}{\varepsilon}\|U\|_{W_\varepsilon^4}$, we lose one power of ε, which seems not to be optimal considering for instance the fast oscillating functions $x_2\mapsto\exp\left(\dfrac{ik_2x_2}{\varepsilon}\right)$. A challenging question in order to apply semiclassical methods to nonlinear problems is to determine appropriate functional spaces which measures on the one hand the Sobolev regularity of the amplitudes, and on the other hand the oscillation frequency.*

We are finally able **to obtain a uniform life span** for the weakly nonlinear system, thus proving result (1a) of Theorem 6. Indeed combining the trilinear estimate (5.4) and the commutator estimate (5.2), we obtain the following Gronwall inequality

$$\varepsilon^2\frac{d}{dt}\left(\|D_\varepsilon^2\tilde{U}\|_{L^2}^2+\|(\varepsilon\partial_1)^4\tilde{U}\|_{L^2}^2+\|\tilde{U}\|_{L^2}^2\right)\leq C\varepsilon^2\left(1+\|D_\varepsilon^2\tilde{U}\|_{L^2}+\|(\varepsilon\partial_1)^4\tilde{U}\|_{L^2}+\|\tilde{U}\|_{L^2}\right)^3$$

from which we deduce the uniform a priori estimate

$$\|D_\varepsilon^2\tilde U\|_{L^2}^2+\|(\varepsilon\partial_1)^4\tilde U\|_{L^2}^2+\|\tilde U\|_{L^2}^2\leq (C_0-Ct)^{-2}$$

where C_0 depends only on the initial data. Such an estimate shows that the life span of the solutions is at least $T^*=C_0/C$.

Finally let us consider the approximation by the nonlinear dynamics, and prove results (1b) and (2) of Theorem 6. The proof of both results relies on standard energy estimates.

If $\eta=0$ and $\varepsilon V_\varepsilon\to 0$ in L^∞, we use the decomposition

$$\varepsilon^2\partial_t(U_\varepsilon-V_\varepsilon)+A(x,\varepsilon D_x)(U_\varepsilon-V_\varepsilon)+\varepsilon^3(\tilde S_j(U_\varepsilon)-\tilde S_j(V_\varepsilon))\varepsilon\partial_jU_\varepsilon+\varepsilon^3\tilde S_j(V_\varepsilon)\varepsilon\partial_jU_\varepsilon=0$$

and obtain the following L^2 estimate

$$\begin{aligned}\frac{\varepsilon^2}{2}\frac{d}{dt}\|U_\varepsilon-V_\varepsilon\|_{L^2}^2&\leq 3\varepsilon^3\|\varepsilon\partial_jU_\varepsilon\|_{L^\infty}\|U_\varepsilon-V_\varepsilon\|_{L^2}^2+3\varepsilon^3\|V_\varepsilon\|_{L^\infty}\|\varepsilon\partial_jU_\varepsilon\|_{L^2}\|U_\varepsilon-V_\varepsilon\|_{L^2}\\&\leq C\varepsilon^2(\varepsilon\|\varepsilon\partial_jU_\varepsilon\|_{L^\infty}+\|\varepsilon\partial_jU_\varepsilon\|_{L^2}^2)\|U_\varepsilon-V_\varepsilon\|_{L^2}^2+C\varepsilon^2(\varepsilon\|V_\varepsilon\|_{L^\infty})^2\end{aligned}$$

from which we conclude by Gronwall's lemma

$$\|U_\varepsilon-V_\varepsilon\|_{L^2}^2\leq C\int_0^t(\varepsilon\|V_\varepsilon(s)\|_{L^\infty})^2\exp C\left(\int_s^t(\varepsilon\|\varepsilon\partial_jU_\varepsilon\|_{L^\infty}+\|\varepsilon\partial_jU_\varepsilon\|_{L^2}^2)d\sigma\right)ds$$

on $[0,T^*[$, and that proves result (1b).

If $\eta>0$, the same arguments show that the life span of the solutions to (5.1) tends to infinity as $\varepsilon\to 0$: $T_\varepsilon\geq C\varepsilon^{-\eta}$, and that these solutions are uniformly bounded in W_ε^4 on any finite time interval. Furthermore, on any finite time interval $[0,T]$, the previous energy estimate gives

$$\frac{\varepsilon^2}{2}\frac{d}{dt}\|U_\varepsilon-V_\varepsilon\|_{L^2}^2\leq C\varepsilon^{3+\eta}\|\varepsilon\partial_jU_\varepsilon\|_{L^\infty}\|U_\varepsilon\|_{L^2}^2\,,$$

from which we deduce

$$\|U_\varepsilon-V_\varepsilon\|_{L^2}^2\leq C\varepsilon^\eta\int_0^t\varepsilon\|\varepsilon\partial_jU_\varepsilon(s)\|_{L^\infty}\|U_\varepsilon(s)\|_{L^2}^2ds\,.$$

Result (2) of Theorem 6 is proved, and that ends the proof of Theorem 6. □

Appendix A. Some well-known facts in semi-classical analysis

In this section we recollect some well-known facts in semi-classical analysis, which have been used throughout the paper. Most of the material is taken from [6], [14], [22] and [23].

A.1. **Semi-classical symbols and operators.**

A.1.1. *Definitions.* We recall that an **order function** is any function $g \in C^\infty(\mathbf{R}^d;\mathbf{R}^+ \setminus \{0\})$ such that there is a constant C satisfying

$$\forall X \in \mathbf{R}^d,\ \forall \alpha \in \mathbf{N}^d, \quad |\partial^\alpha g(X)| \leq Cg(X).$$

For instance $g(x,\xi) = (1+|\xi|^2)^{\frac{1}{2}} =: \langle \xi \rangle$ is an order function. Note that the variable X usually refers to a point (x,ξ) in the cotangent space $T^*\mathbf{R}^n \equiv \mathbf{R}^{2n}$, or to a point of the type (x,y,ξ) with $y \in \mathbf{R}^n$. A **semi-classical symbol** in the class $S_d(g)$ is then a function $a = a(X;\varepsilon)$ defined on $\mathbf{R}^d \times]0,\varepsilon_0]$ for some $\varepsilon_0 > 0$, which depends smoothly on X and such that for any $\alpha \in \mathbf{N}^d$, there is a constant C such that $|\partial^\alpha a(X,\varepsilon)| \leq Cg(X)$ for any $(X,\varepsilon) \in \mathbf{R}^d \times]0,\varepsilon_0]$.

If $(a_j)_{j\in\mathbf{N}}$ is a family of semi-classical symbols in the class $S_d(g)$, we write that

$$a = \sum_{j=0}^{\infty} \varepsilon^j a_j + O(\varepsilon^\infty)$$

if for any $N \in \mathbf{N}$ and for any $\alpha \in \mathbf{N}^d$, there are ε_0 and C such that

$$\forall X \in \mathbf{R}^d,\ \forall \varepsilon \in]0,\varepsilon_0] \quad \Big|\partial^\alpha \Big(a(X,\varepsilon) - \sum_{j=0}^{N} \varepsilon^j a_j(X,\varepsilon)\Big)\Big| \leq C\varepsilon^N g(X).$$

Conversely for any sequence $(a_j)_{j\in\mathbf{N}}$ of symbols in $S_d(g)$, there is $a \in S_d(g)$ (unique up to $O(\varepsilon^\infty)$) such that $a = \sum_{j=0}^{\infty} \varepsilon^j a_j + O(\varepsilon^\infty)$. An **$\varepsilon$-pseudodifferential operator** is defined as follows: if a belongs to $S_{3n}(g)$, and u is in $\mathcal{D}(\mathbf{R}^n)$, then

$$\Big(\mathrm{Op}_\varepsilon(a)\Big)u(x) := \frac{1}{(2\pi\varepsilon)^n} \int e^{i(x-y)\cdot\xi/\varepsilon} a(x,y,\xi)u(y)\, dyd\xi.$$

A.1.2. *Changes of quantization.* If $a \in S_{2n}(g)$ and $t \in [0,1]$ then $a^t(x,y,\xi) := a((1-t)x+ty,\xi)$ belongs to $S_{3n}(g)$, and one defines $\mathrm{Op}_\varepsilon^t(a) := \mathrm{Op}_\varepsilon(a^t)$. When $t=0$ this corresponds to the **classical**, or "**left**" quantization, and when $t = 1/2$ this is known as the **Weyl quantization** (and is usually denoted by $\mathrm{Op}_\varepsilon^W(a) = \mathrm{Op}_\varepsilon^{\frac{1}{2}}(a)$).

A **classical symbol** is a symbol a in $S_{2n}(\langle\xi\rangle^m)$ such that

$$a(x,\xi;\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j a_j(x,\xi) + O(\varepsilon^\infty)$$

with a_0 not identically zero, and $a_j \in S_{2n}(\langle\xi\rangle^m)$ independent of ε. The term $\varepsilon^\nu a_0$ is the **principal symbol** of the classical pseudo-differential operator $A = \mathrm{Op}_\varepsilon^t(a)$ (and this does not depend on the quantization). On the other hand a_1 is the **subprincipal** symbol of $A = \varepsilon^\nu \mathrm{Op}_\varepsilon^W(a)$ (in the Weyl quantization only). In the following we shall denote by $\sigma_t(A)$ the symbol of an operator $A = \mathrm{Op}_\varepsilon^t(a)$ (in other words $a = \sigma_t(A)$), and by $\sigma_P(A)$ its principal symbol.

A.1.3. *Microlocal support and ε-oscillation.* If u is an ε-dependent function in a ball of $L^2(\mathbf{R}^n)$, its **ε-frequency set** (or **microlocal support**) is the complement in $\mathbf{R}^{2n}$ of the points (x_0,ξ_0) such that there is a function $\chi_0\in S_{2n}(1)$ equal to one at (x_0,ξ_0), satisfying

$$\|\mathrm{Op}_\varepsilon^W(\chi_0 u)\|_{L^2(\mathbf{R}^{2n})}=O(\varepsilon^\infty).$$

We say that an ε-dependent function f_ε bounded in $L^2(\mathbf{R}^n)$ is **ε-oscillatory** if for every continuous, compactly supported function φ on $\mathbf{R}^n$,

$$\limsup_{\varepsilon\to 0}\int_{|\xi|\geq R/\varepsilon}|\varphi\widehat{f}_\varepsilon(\xi)|^2\,d\xi\to 0\quad\text{as}\quad R\to\infty. \tag{A.1}$$

An ε-dependent function f_ε bounded in $L^2(\mathbf{R}^n)$ is said to be **compact at infinity** if

$$\limsup_{\varepsilon\to 0}\int_{x\geq R}|f_\varepsilon(x)|^2\,dx\to 0\quad\text{as}\quad R\to\infty. \tag{A.2}$$

A.1.4. *Adjoint and composition.* Let a be a symbol in $S_{3n}(g)$, and define $a^*(x,y,\xi):=\overline{a(y,x,\xi)}$. Then the operator $(\mathrm{Op}_\varepsilon(a))^*:=\mathrm{Op}_\varepsilon(a^*)$ satisfies for all u,v in $\mathcal{S}(\mathbf{R}^n)$,

$$\Big((\mathrm{Op}_\varepsilon(a))^*u,v\Big)_{L^2}=\Big(u,(\mathrm{Op}_\varepsilon(a))^*v\Big)_{L^2}$$

and is therefore called the **formal adjoint** of $\mathrm{Op}_\varepsilon(a)$. In particular $\mathrm{Op}_\varepsilon^W(a)$ is **formally self-adjoint** if a is real. Let a and b be two symbols in $S_{2n}(g_1)$ and $S_{2n}(g_2)$ respectively. For all $t\in[0,1]$, there is a unique symbol c_t in $S_{2n}(g_1g_2)$ which allows to obtain $\mathrm{Op}_\varepsilon^t(a)\circ\mathrm{Op}_\varepsilon^t(b)=\mathrm{Op}_\varepsilon^t(c_t)$. Moreover one has

$$c_t(x,\xi;\varepsilon)=e^{i\varepsilon[\partial_u\partial_\xi-\partial_\eta\partial_v]}\left(a((1-t)x+tu,\eta)b((1-t)v+tx,\xi)\right)_{|\substack{u=v=x\\ \eta=\xi}}=:a\#^t b. \tag{A.3}$$

This can be also written

$$c_t(x,\xi;\varepsilon)=\sum_{k\geq 0}\frac{\varepsilon^k}{i^kk!}(\partial_\eta\partial_v-\partial_\xi\partial_u)^k\left(a((1-t)x+tu,\eta)b((1-t)v+tx,\xi)\right)_{|\substack{u=v=x\\ \eta=\xi}}+O(\varepsilon^\infty).$$

In particular one has $\sigma_t(A\circ B)=\sigma_t(A)\sigma_t(B)+O(\varepsilon)$. Similarly if a and b are two symbols. Then the principal symbol of $\mathrm{Op}_\varepsilon^W(a)\mathrm{Op}_\varepsilon^W(b)$ is ab and its subprincipal symbol is $\frac{1}{2i}\{a,b\}$.

A.2. **Semiclassical operators, Wigner transforms and propagation of energy.** One of the main interests of the semiclassical setting is that it allows a precise description of the propagation of the energy, on times of the order of $O(\varepsilon)$. We refer for instance to [11] (Section 6) for the proof of the following property (actually in the more general setting of matrix-valued operators): consider a scalar symbol $\tau_\varepsilon(x,\xi)$ defined on $\mathbf{R}^{2n}$, belonging to the class $S_{2n}(\langle\xi\rangle^\sigma)$ for some $\sigma\in\mathbf{R}$ (or more generally to $S_{2n}(g)$). We assume moreover that $\mathrm{Op}_\varepsilon^W(\tau_\varepsilon)$ is essentially skew-self-adjoint on $L^2(\mathbf{R}^n)$. Then consider f_ε^0 an ε-oscillatory initial data in the sense of (A.1), bounded in $L^2(\mathbf{R}^n)$ and compact at infinity in the sense of (A.2), and the PDE

$$\varepsilon\partial_t f_\varepsilon+\mathrm{Op}_\varepsilon^W(\tau_\varepsilon)f_\varepsilon=0,\quad f_{\varepsilon|t=0}=f_\varepsilon^0.$$

Then the Wigner transform $W_\varepsilon(t,x,\xi)$ of $f_\varepsilon(t)$ defined by

$$W_\varepsilon(t,x,\xi):=(2\pi)^{-n}\int_{\mathbf{R}^n}e^{iv\cdot\xi}f_\varepsilon(x-\frac{\varepsilon}{2}v)\bar{f}_\varepsilon(x+\frac{\varepsilon}{2}v)\,dv$$

converges, locally uniformly in t, to the solution W of $\partial_t W + \{\tau_0, W\} = 0$ where τ_0 is the principal symbol of τ_ε, and where the Poisson bracket is given by

$$\{\tau_0, W\} := \nabla_\xi \tau_0 \cdot \nabla_x W - \nabla_x \tau_0 \cdot \nabla_\xi W.$$

The interest of Wigner transforms lies in particular in the fact that under the assumptions made on f^0_ε, for any compact set $K \subset \mathbf{R}^n$ one has $\displaystyle\int_K |f_\varepsilon(t,x)|^2\, dx = W_\varepsilon(t, K \times \mathbf{R}^n)$ due to the fact that $|f_\varepsilon(t,x)|^2 = \displaystyle\int_{\mathbf{R}^n} W_\varepsilon(t,x,\xi)\, d\xi$.

Finally let us recall that if $f \in L^2(\mathbf{R}^n)$ of norm 1 is a solution to

$$\mathrm{Op}^W(p) f = 0$$

where p is a classical symbol of principal part p_0, then the microlocal support of f is included in the chararteristic set

$$\left\{(x,\xi) \in \mathbf{R}^{2n}\, , p_0(x,\xi) = 0\right\}.$$

A.3. **Coherent states.** A **coherent state** is $\Phi_{p,q}(y) := (\pi\varepsilon)^{-\frac{n}{4}} e^{i\frac{(y-q)\cdot p}{\varepsilon}}\ e^{-\frac{(y-q)^2}{2\varepsilon}}$. Any tempered distribution u defined on $\mathbf{R}^n$ may be written

$$u(y) = (2\pi\varepsilon)^{-\frac{n}{2}} \int Tu(p,q) \Phi_{p,q}(y)\, dpdq,$$

where T is the FBI (for Fourier-Bros-Iagolnitzer) transform

$$Tu(p,q) := 2^{-\frac{n}{2}} (\pi\varepsilon)^{-\frac{3n}{4}} \int e^{i\frac{(q-y)\cdot p}{\varepsilon}}\ e^{-\frac{(y-q)^2}{2\varepsilon}} u(y)\, dy.$$

This transformation maps isometrically $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{2n})$. The above formula simply translates the fact that $u = T^*Tu$.

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